Date ⇒ 04 · 10 · 23

Exercise - 7 (l)

1. If $y = \tan^{-1} x$ prove that

$$(1+x^2)\, y_2 + 2xy_1 = 0$$

Ans ⇒ $y = \tan^{-1} x \Rightarrow y_1 = \dfrac{1}{1+x^2}$

$y_1 = \dfrac{dy}{dx}$

$y_2 = \dfrac{d}{dx}(\ )$

$\dfrac{dy}{dx}$

⇒ $(1+x^2)\, y_1 = 1$

$y_2 = \frac{dy_1}{dx}$

⇒ $(1+x^2)\, y_2 + 2xy_1 = 0$

(2) If $2y = x\left(1+\frac{dy}{dx}\right)$ show that y_2 is a constant

Ans ⇒ $2y = x\left(1+\frac{dy}{dx}\right)$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

⇒ $2\frac{dy}{dx} = \left(1+\frac{dy}{dx}\right) \cdot 1 + x\frac{d^2y}{dx^2}$

⇒ $x\frac{d^2y}{dx^2} = \frac{dy}{dx} - 1$

⇒ $\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}-1}{x}$

⇒ $\frac{d^3y}{dx^3} = \frac{x\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}-1\right)}{x^2} = 0$ $\left[\frac{dy}{dx}-1 = x\frac{d^2y}{dx^2} \text{ By } (1)\right]$

∴ $\frac{d^2y}{dx^2}$ is constant i.e. y_2 is a constant

(2) If $y = ax\sin x$ show that

$x^2y_2 - 2xy_1 + (x^2+2)\,y = 0$

Ans ⇒ $y = ax\sin x$

⇒ $y_1 = a\sin x + ax\cos x$

⇒ $y_2 = a\cos x + a\cos x - ax\sin x$

$= 2a\cos x - ax\sin x$

Now $x^2y_2 - 2xy_1 + (x^2+2)y$

$= 2ax^2\cos x - ax^3\sin x - 2ax\sin x - 2ax^2\cos x + ax^3\sin x + 2ax\sin x = 0$

(4) If $y = e^{m\cos^{-1}x}$

$(1-x^2)y_2 - xy_1 = m^2y$

Ans: $y = e^{m\cos^{-1}x}$

$\Rightarrow \ln y = m\cos^{-1}x$

$\Rightarrow \frac{1}{y}\cdot y_1 = \frac{-m}{\sqrt{1-x^2}}$

$\Rightarrow (\sqrt{1-x^2})\cdot y_1 = -my$

$\Rightarrow (1-x^2)(y_1)^2 = m^2y^2$

$\Rightarrow (1-x^2)\cdot 2y_1y_2 - 2x(y_1)^2 = 2m^2yy_1$

$\Rightarrow (1-x^2)y_2 - xy_1 = m^2y$

(5) If $x = \sin t$, $y = \sin 2t$ then prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$

Ans: $x = \sin t$, $y = \sin 2t$

$\Rightarrow \frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = 2\cos 2t$

$\Rightarrow \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2\cos 2t}{\cos t}$

$= \frac{2\sqrt{1-\sin^2 2t}}{\sqrt{1-\sin^2 t}}$

$= \frac{2\sqrt{1-y^2}}{\sqrt{1-x^2}} = 2\sqrt{\frac{1-y^2}{1-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{\frac{1-y^2}{1-x^2}}\right)^2$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4\left(\frac{1-y^2}{1-x^2}\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 (1-x^2) = 4(1-y^2)$

$\Rightarrow 2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}(1-x^2) + \left(\frac{dy}{dx}\right)^2(-2x)$

$= 4 \cdot -2y\frac{dy}{dx}$

$\Rightarrow 2\left(\frac{dy}{dx}\right)\cdot\frac{d^2y}{dx^2}(1-x^2) - 2x\left(\frac{dy}{dx}\right)^2 + 4\cdot 2y\left(\frac{dy}{dx}\right) = 0$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$

(6) If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)\,y_2 - xy_1 - 2 = 0$

Ans ⇒ $y = (\sin^{-1}x)^2$

$\Rightarrow y_1 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\,y_1 = 2\sin^{-1}x$

$\Rightarrow (1-x^2)(y_1)^2 = 4(\sin^{-1}x)^2 = 4y$

$\Rightarrow (1-x^2)\,2y_1y_2 - 2x(y_1)^2 = 4y_1$

$\Rightarrow (1-x^2)\,y_2 - xy_1 - 2 = 0$

(7) If $y = \tan^{-1}x$, prove that $(1+x^2)y_2 + 2xy_1 = 0$

Ans ⇒ same as No. 2