Date :- 26.11.23

Integration by parts :-

Int. of product = (Int. first) × second

$- \int$ (Int. first) (der. second) dx

$\boxed{\text{ETALI}}$

ଏହି ଅନୁସାରେ first and second function choose କରାଯିବ।

E – Exponential function (e^x)

T – Trigonometric function ($\sin x$, $\cos x$)

A – Algebric function (x^n, 1. $1 = x^0$)

L – Logarithmic function ($\ln x$)

I – Inverse trigonometric function ($\cos^{-1}x$, $\sin^{-1}x$)

function to be integration	first function	second function
$x^n e^x$	e^x	x^n
$x^n \cos x$	$\cos x$	x^n
$x^n \sin x$	$\sin x$	x^n
$x^n (\ln x)^m$	x^n	$(\ln x)^m$
$x^n \sin^{-1}x$	x^n	$\sin^{-1}x$
$x^n \cos^{-1}x$	x^n	$\cos^{-1}x$
$x^n \tan^{-1}x$	x^n	$\tan^{-1}x$

Date → 17/11/2023 Exercise - 9(e)

1(i) $\int (1+x) e^{x} dx$

$= \left(\int e^{x} dx\right)(1+x) - \int \left(\int e^{x} dx\right) \frac{d}{dx}(1+x) dx$

$= e^{x}(1+x) - \int e^{x} dx$

$= e^{x} + xe^{x} - e^{x} + c$

$= xe^{x} + c$

(ii) $\int x^3 e^{x} dx$

$= \left(\int e^{x} dx\right) x^3 - \int \left(\int e^{x} dx\right) \frac{d}{dx} x^3 dx$

$= x^3 e^{x} - \int e^{x} 3x^2 dx$

$= x^3 e^{x} - 3\left[\int e^{x} x^2 dx\right]$

$= x^3 e^{x} - 3\left[\int e^{x} dx . x^2 - \int e^{x} dx \left(\frac{dx^2}{dx}\right) dx\right]$

$= x^3 e^{x} - 3\left[x^2 e^{x} - \int 2x e^{x} dx\right]$

$= x^3 e^{x} - 3\left[x^2 . e^{x} - 2\left[\int e^{x} dx . x - \int e^{x} dx \left(\frac{dx}{dx}\right) dx\right]\right.$

$= x^3 e^{x} - 3\left[x^2 e^{x} - 2\left[x e^{x} - e^{x}\right]\right]$

$= x^3 e^{x} - 3x^2 e^{x} + 6x e^{x} - 6e^{x}$

$= e^{x}\left[x^3 - 3x^2 + 6x - 6\right]$

(iii) $\int x^2 e^{ax} dx$

$$= \int e^{ax}dx \cdot x^2 - \int \left(\int e^{ax}dx\right)\frac{dx^2}{dx}dx$$

$$= \frac{1}{a}e^{ax}x^2 - \int \frac{1}{a}e^{ax}\cdot 2x\,dx$$

$$= \frac{e^{ax}}{a}\cdot x^2 - \int \frac{e^{ax}}{a}2x\,dx$$

$$= \frac{e^{ax}}{a}x^2 - \frac{2}{a}\int e^{ax}x\,dx$$

$$= \frac{e^{ax}}{a}x^2 - \frac{2}{a}\left[\int e^{ax}dx \cdot x - \int\left(\int e^{ax}\frac{dx}{dx}dx\right)\right]$$

$$= \frac{e^{ax}}{a}x^2 - \frac{2}{a}\left[\frac{1}{a}e^{ax}x - \int \frac{e^{ax}}{a}dx\right]$$

$$= \frac{e^{ax}}{a}x^2 - \frac{2}{a}\left[\frac{1}{a}e^{ax}x - \frac{1}{a}\cdot\frac{1}{a}e^{ax}\right]$$

$$= \frac{e^{ax}}{a}x^2 - \frac{2}{a^2}e^{ax}x + \frac{2}{a^3}e^{ax} + c$$

$$= \frac{e^{ax}}{a}\left[x^2 - \frac{2x}{a} + \frac{2}{a^2}\right] + c$$

(iv) $\int (3x+2)^2 e^{2x}dx$

$$= \int e^{2x}dx\,(3x+2)^2 - \int\left(\int e^{2x}\frac{d(3x+2)^2}{dx}\cdot dx\right)$$

$$= \frac{1}{2}e^{2x}(3x+2)^2 - \int \frac{e^{2x}}{\cancel{2}}\cancel{2}(3x+2)3\,dx$$

$$= \frac{e^{2x}}{2}(3x+2)^2 - \left[3\int e^{2x}(3x+2)dx\right]$$

$$= \frac{e^{2x}}{2}(3x+2)^2 - 3\left[\int e^{2x}\cdot(3x+2)dx - \int\left(\int e^{2x}\frac{d(3x+2)}{dx}dx\right)\right]$$

$= \frac{e^{2x}}{2}(3x+2)^2 - 3\left[e^{2x}(3x+2) - \int \frac{e^{2x}}{2}\cdot 3\,dx\right]$

$= \frac{e^{2x}}{2}(3x+2)^2 - \frac{3}{2}e^{2x}(3x+2) + 9\cdot\frac{1}{2}e^{2x}+c$

$= \frac{e^{2x}}{2}\left[(3x+2)^2 - 3(3x+2) + \frac{9}{2}\right]+c$

2.(i) $\int x\sin x\,dx$

$= \int \sin x\,dx\cdot x - \int\left(\int \sin x\cdot\frac{dx}{dx}\,dx\right)$

$= -x\cos x + \int\cos x + c$

$= -x\cos x + \sin x + c$

$= \sin x - x\cos x + c$

(ii) $\int x^2\cos x\,dx$

$= \int \cos x\,dx\,x^2 - \int\left(\int\cos x\cdot\frac{dx^2}{dx}\cdot dx\right)$

$= x^2\sin x - \int 2x\sin x\,dx$

$= x^2\sin x - 2\left[\int\sin x\cdot x - \int\left(\int\sin x\cdot\frac{dx}{dx}\,dx\right)\right]$

$= x^2\sin x + 2x\cos x + 2\int\cos x + c$

$= x^2\sin x + 2x\cos x - 2\sin x + c$

(iii) $\int x^2\sin ax\,dx$

$= \int \sin ax\,dx \cdot x^2 - \int \left(\int \sin ax \cdot \frac{dx^2}{dx}\right) dx$

$= \frac{1}{a} - \cos ax \cdot x^2 - \int \frac{1}{a} - \cos ax \cdot 2x\,dx$

$= -\frac{x^2}{a}\cos ax + \frac{2}{a}\int \cos ax \cdot x\,dx$

$= -\frac{x^2}{a}\cos ax + \frac{2}{a}\left[x\int \cos ax\,dx - \int\left(\cos ax\,\frac{dx}{dx}\right) dx\right]$

$= -\frac{x^2}{a}\cos ax + \frac{2}{a}\cdot\frac{x}{a}\sin ax + \frac{2}{a}\cdot\frac{1}{a}\sin ax$

$= -\frac{x^2}{a}\cos ax + \frac{2x}{a^2}\sin ax + \frac{2}{a^2}\,\frac{1}{a}\cos ax + c$

$= -\frac{x^2}{a}\cos ax + \frac{2x}{a^2}\sin ax + \frac{2}{a^3}\cos ax + c$

(iv) $\int x \cos^2 x\,dx$

$= \int x \frac{(1+\cos 2x)}{2}\,dx$

$= \frac{1}{2}\int (x + x\cos 2x)\,dx$

$= \frac{1}{2}\left[\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx\right]$

$= \frac{1}{2}\left[\frac{x^2}{2} + \int \cos 2x\,dx \cdot x - \int\left(\int \cos 2x\,dx \cdot \frac{dx}{dx}\right)dx\right]$

$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx\right]$

$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} + \frac{1}{2}\,\frac{\cos 2x}{2}\right]$

$= \frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + c$

(v) $\int x \sin^3 x\,dx$ $\quad \left[\begin{array}{l} \text{Let } \cos x = t \\ \sin x\,dx = dt \end{array}\right.$

$= \int \sin^3 x\,dx \cdot x - \int \left(\int \sin^3 x \cdot \frac{dx}{dx} \cdot dx \right)$

$= \int (1-\cos^2 x) \sin x\,dx \cdot x - \int \left(\int (1-\cos^2 x) \sin x\,dx \right)$

$= \int (1-t^2) - dt \cdot x - \int \left(\int (1-t^2)\,dt \right)$

$= \int -1\,dt\,x + \int t^2\,dt\,x - \int \int (1-t^2)\,dt$

$= -x\cos x + \frac{x\cos^3 x}{3} - \left[\int \int 1\,dt - \int t^2\,dt \right]$

$= -x\cos x + \frac{x\cos^3 x}{3} - \left[\int \cos x - \int \frac{\cos^3 x}{3}\,dx \right]$

$= -x\cos x + \frac{x\cos^3 x}{3} - \sin x + \frac{1}{3} \int (1-\sin^2 x) \cos x\,dx$

$= -x\cos x + \frac{x\cos^3 x}{3} - \sin x + \frac{1}{3} \int (1-z^2)\,dz$

$= -x\cos x + \frac{x\cos^3 x}{3} - \sin x + \frac{1}{3}\sin x$ $\quad \therefore \text{Let } \sin x = z$

$-\frac{1}{9}\sin^3 x + C$ $\quad \cos x\,dx = dz$

(vi) $\int 2x \sin 2x \cos x\,dx$

$= \int (2x \sin 2x) \cos x\,dx$

$= \left[\int \sin 2x\,dx\,2x - \int \left(\int \sin 2x \frac{d\,2x}{dx} \cdot dx \right) \right] (\cos x\,dx)$

$= \left[\frac{1}{\cancel{2}} - \cos 2x \cdot \cancel{2}x - \int 2 \cdot \frac{1}{2} - \cos 2x \right] (\cos x\,dx)$

$= \left[-\cos 2x \cdot x + \int \cos 2x \right] (\cos x\,dx)$

$= \int \left(-\cos 2x + \frac{1}{2}\sin 2x \right) \cos x\,dx$

$= \int \cos x - \cos 2x\, dx + \int \frac{1}{2} \sin 2x \cos x\, dx$

(vi) $\int 2x \sin 2x \cos x\, dx$

$= 2\int x \sin 2x \cdot \cos x\, dx$

$= 2\int x \frac{1}{2} \times 2 \sin 2x \cos x\, dx$

$= \frac{2}{2}\int x (\sin 2x + \sin x)\, dx$

$= \int x (\sin 3x + \sin x)\, dx$

$= \int x \sin 3x\, dx + \int x \sin x\, dx$

$= \int \sin 3x\, dx \cdot x - \int \left(\int \sin 3x\, dx \frac{dx}{dx}\right) dx$

$+ \int \sin x\, dx \cdot x - \int \left(\int \sin x\, dx \frac{dx}{dx}\right)$

$= -\frac{x}{3} \cos 3x - \int -\frac{1}{3} \cos 3x\, dx$

$- x\cos x - \int -\cos x\, dx$

$= -\frac{x}{3} \cos 3x + \frac{1}{3} \cdot \frac{1}{3} \sin 3x - x\cos x$

$+ \sin x$

$= -\frac{x}{3} \cos 3x + \frac{1}{9} \sin 3x - x\cos x$

$+ \sin x + c$

(viii) $\int 2x^3 \cos x^2\, dx$

$= \int 2x \cdot x^2 \cos x^2\, dx$

$= \int t\, dt \quad \cos t$

$= \int t \cos t\, dt$

$$= \int \cos t\, dt \cdot t - \int \left(\int \cos t \cdot \frac{dt}{dt} \right) dt$$

$$= t \sin t - \int \sin t \, ; \quad = t\sin t + \cos t + c$$

$$= x^2 \sin x^2 + \cos x^2 + c$$

(vii) $\int 2x \cos 3x \cos 2x \, dx$

$$= 2\int x \cos 3x \cos 2x \, dx$$

$$= 2\int x \left(\frac{1}{2} \times 2 \cos 3x \cos 2x \right) dx$$

$$= \frac{2}{2} \int x (\cos 3x + \cos 2x) \, dx$$

$$= \int x (\cos 5x + \cos x) \, dx$$

$$= \int x \cos 5x + \int x \cos x \, dx$$

$$= \int \cos 5x \, dx \cdot x - \int \left(\int \cos 5x \cdot \frac{dx}{dx} \right) dx$$

$$+ \int \cos x \, dx \cdot x - \int \left(\int \cos x \frac{dx}{dx} \right) dx$$

$$= \frac{x}{5} \sin 5x - \int \frac{1}{5} \sin 5x$$

$$+ x \sin x - \int \sin x \, dx$$

$$= \frac{x}{5} \sin 5x + \frac{1}{5} \cdot \frac{1}{5} \cos 5x$$

$$+ x \sin x + \cos x + c$$

$$= \frac{1}{5} x \sin 5x + \frac{1}{25} \cos 5x + x \sin x$$

$$+ \cos x + c$$

(ix) $\int x \csc^2 x \, dx$

$= \int \csc^2 x \, dx \cdot x - \int \left(\int \csc^2 x \cdot \frac{dx}{dx} \right) \cdot dx$

$= -x\cot x - \int -\cot x \, dx$

$= -x\cot x + \ln|\sin x| + c$

(x) $\int x \tan^2 x \, dx$

$= \int x(\sec^2 x - 1)\, dx$

$= \int x \sec^2 x \, dx - \int x \, dx$

$= \int \sec^2 x \, dx \cdot x - \int \left(\int \sec^2 x \, \frac{dx}{dx} \right) \cdot dx - \frac{x^2}{2}$

$= x\tan x - \int \tan x \, dx - \frac{x^2}{2}$

$= x\tan x + \ln|\sec x| - \frac{x^2}{2} + c$

3 (i) $\int x \ln(1+x)\, dx$

$= \int x \, dx \ln(1+x) - \int \left(\int x \, dx \, \frac{d\ln(1+x)}{dx} \right)$

$= \frac{x^2}{2}\ln(1+x) - \int \frac{x^2}{2} \cdot \frac{1}{1+x}\, dx$

$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int \frac{x^2}{1+x}\, dx$

$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2} \frac{x^2 - 1 + 1}{1+x}\, dx$

$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\left\{ \int \frac{x^2-1}{1+x}\, dx + \int \frac{1}{1+x}\, dx \right.$

$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int (x-1)\, dx - \frac{1}{2}\int \frac{1}{1+x}\, dx$

$= \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(1+x) + C$

(ii) $\int x^7 \ln x\,dx$

$= \int x^7 dx \ln x - \int\left(\int x^7 \cdot dx \cdot \frac{d \ln x}{dx}\right)dx$

$= \frac{x^8}{8}\ln x - \int \frac{x^8}{8}\cdot\frac{1}{x}$

$= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7\,dx$

$= \cancel{\frac{1}{8}\left(x^8 \ln x - \frac{x^8}{8}\right)} = \frac{1}{8}x^8 \ln x - \frac{1}{8}\cdot\frac{x^8}{8} + C$

$= \frac{1}{8}x^8 \ln x - \frac{x^8}{64}$

$= \frac{x^8}{8}\left(\ln x - \frac{1}{8}\right) + C$

(iii) $\int (\ln x)^3 dx = \int 1\,(\ln x)^3 dx$

$= \int 1\,dx\cdot(\ln x)^3 - \int\left(\int dx \cdot \frac{d(\ln x)^3}{dx}\right)$

$= x(\ln x)^3 - \int \cancel{x}\cdot 3(\ln x)^2\cdot\frac{1}{\cancel{x}}$

$= x(\ln x)^3 - 3\left[\int 1\,dx\cdot(\ln x)^2 - \int\left(\int 1\,dx\cdot\frac{d(\ln x)^2}{dx}\right)\right]$

$= x(\ln x)^3 - 3\left[x(\ln x)^2 - \int \cancel{x}\,2\ln x\cdot\frac{1}{\cancel{x}}\right]$

$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int \ln x\,dx\right]$

$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\left[\int 1\,dx\cdot\ln x - \int\left(\int 1\,dx\cdot\frac{d\ln x}{dx}\right)\right.\right.$

$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int x \cdot \frac{dx}{x}$

$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + c$

$= x\left[(\ln x)^3 - 3(\ln x)^2 + 6(\ln x) - 6\right] + c$

(iv) $\int \ln(x^2+1)\,dx = \int 1 \cdot \ln(x^2+1)\,dx$

$= \int 1\,dx \ln(x^2+1) - \int\left(\int 1\,dx \cdot \frac{d\ln(x^2+1)}{dx}\right)dx$

$= x\ln(x^2+1) - \int x \cdot \frac{2x}{x^2+1}\,dx$

$= x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx$

$= x\ln(x^2+1) - \int \frac{2x^2+2-2}{x^2+1}\,dx$

$= x\ln(x^2+1) - \left[\int \frac{2x^2+2}{x^2+1}\,dx - \int \frac{2}{x^2+1}\,dx\right]$

$= x\ln(x^2+1) - \left[\int \frac{2(x^2+1)}{(x^2+1)}\,dx - 2\int \frac{1}{x^2+1}\,dx\right]$

$= x\ln(x^2+1) - \left[\int 2\,dx - 2\int \frac{dx}{x^2+1}\,dx\right]$

$= x\ln(x^2+1) - 2x + 2\,\frac{1}{1}\tan^{-1}\frac{x}{1} + c$

$= x\ln(x^2+1) - 2x + 2\tan^{-1}x + c$

(v) $\int \frac{\ln x}{x^5}\,dx$

$= \int x^{-5}\ln x\,dx$

$= \int x^{-5}dx \ln x - \int\left(\int x^{-5}\,\frac{d\ln x}{dx}\right)dx$

$= \frac{x^{-4}}{-4}\ln x - \int \frac{x^{-4}}{-4}\cdot\frac{1}{x}\,dx$

$= -\frac{1}{4}x^{-4}\ln x - \int \frac{x^{-5}}{-4}\,dx$

$= -\frac{1}{4}x^{-4}\ln x + \frac{1}{4}\frac{x^{-4}}{-4} + c$

$= -\frac{1}{4}x^{-4}\ln x - \frac{1}{16}x^{-4} + c$

$= -\frac{1}{4}x^{-4}\left(\ln x + \frac{1}{4}\right) + c$

(vi) $\int \ln(x^2+x+2)\,dx$

$= \int 1\,dx\,\ln(x^2+x+2) - \int\left(\int 1\,dx\cdot\frac{d\ln(x^2+x+2)}{dx}\right)dx$

$= x\ln(x^2+x+2) - \int x\cdot\frac{2x+1}{x^2+x+2}\,dx$

$= x\ln(x^2+x+2) - \int\frac{2x^2+x}{x^2+x+2}\,dx$

$= x\ln(x^2+x+2) - \int\frac{2x^2+2x-x+4-4}{x^2+x+2}\,dx$

$= x\ln(x^2+x+2) - \int\frac{2(x^2+x+2)}{x^2+x+2} - \int\frac{(+x-4)}{x^2+x+2}\,dx$

$= x\ln(x^2+x+2) - \int 2\,dx + \int\frac{x+4}{x^2+x+2}\,dx$

$= x\ln(x^2+x+2) - 2x\left[\int\frac{x}{x^2+x+2}\,dx + \int\frac{4}{x^2+x+2}\,dx\right.$

$= x\ln(x^2+x+2) - 2\left[\int\frac{x\,dx}{x^2+x+2} + 4\int\frac{dx}{x^2+x+2}\right]$

$= x\ln(x^2+x+2) - 2[$

Let $x^2+x+2=t$

$2x+1=\frac{dt}{dx}$

$dx=\frac{dt}{2x+1}$

(vii) $\int \ln\left(x+\sqrt{x^2+a^2}\right)dx$

$= \int 1\,dx\,\ln\left(x+\sqrt{x^2+a^2}\right) - \int\left(\int 1\,dx\,\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)dx\right)$

$= x\ln\left(x+\sqrt{x^2+a^2}\right) - \int x\cdot\frac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}\,\sqrt{x^2+a^2}}\,dx$

$= x\ln\left(x+\sqrt{x^2+a^2}\right) - \int \frac{x\,dx}{\sqrt{x^2+a^2}}$

$= x\ln\left(x+\sqrt{x^2+a^2}\right) - \int \frac{dt}{2\sqrt{t}}$

$= x\ln\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{2}\int dt^{-\frac{1}{2}}$

$= x\ln\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{-\frac{1}{2}+1}$

$= x\ln\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{2}\,2\sqrt{t}$

$= x\ln\left(x+\sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2}+c$

[Let $x^2+a^2=t$, $2x=\frac{dt}{dx}$, $x\,dx=\frac{dt}{2}$]

iii) $\int \ln(x+\sqrt{x^2-a^2})\,dx$

$= \int 1dx \ln(x+\sqrt{x^2-a^2}) - \int\left(\int 1dx \cdot \frac{d\ln(x+\sqrt{x^2-a^2})}{dx}\right)dx$

$= x\ln(x+\sqrt{x^2-a^2}) - \int x\cdot\frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}\frac{dx}{\sqrt{x^2-a^2}}$

$= x\ln(x+\sqrt{x^2-a^2}) - \int\frac{x\,dx}{\sqrt{x^2-a^2}}$

$= x\ln|x+\sqrt{x^2-a^2}| - \sqrt{x^2-a^2}+c$

i) $\int\sin^{-1}x\,dx$

$=\int 1dx\,\sin^{-1}x - \int\left(\int 1dx \cdot \frac{d\sin^{-1}x}{dx}\right)dx$

$x\sin^{-1}x - \int x\frac{1}{\sqrt{1-x^2}}dx$

$= x\sin^{-1}x - \int\frac{x}{\sqrt{1-x^2}}dx$

$= x\sin^{-1}x - \int\frac{-dt}{2\sqrt{t}}$

$= x\sin^{-1}x + \frac{1}{2}\int dt \cdot {}^{-\frac{1}{2}}$ Let $1-x^2=t$

$= x\sin^{-1}x + \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}}$ $-2x=\frac{dt}{dx}$

$= x\sin^{-1}x+\frac{1}{\cancel{2}}\cdot\cancel{2}\sqrt{t}$ $x\,dx=-\frac{dt}{2}$

$= x\sin^{-1}x+\sqrt{1-x^2}+c$

ii) $\int x\sin^{-1}x\,dx$

$=\int x\,dx\cdot\sin^{-1}x - \int\left(\int x\,dx\cdot\frac{d\sin^{-1}x}{dx}\right)dx$

$= \frac{x^2}{2}\sin^{-1}x - \int \frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}}\cdot x^2$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\sin^{-1}x\cdot\frac{x^3}{3}+c$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{6}\sin^{-1}x\, x^3+c$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\left[\int \frac{1-\cos 2\theta}{2}\right]d\theta$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\left[\int 1\,d\theta - \int \cos 2\theta\, d\theta\right]$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\left[\sin^{-1}x - \frac{1}{\cancel{2}}\cancel{2}\sin\theta\cdot\cos\theta\right]$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\left[\sin^{-1}x - x\cdot\sqrt{1-x^2}\right]$

$\left[\begin{array}{l} \text{Let } x = \sin\theta \\ 1 = \cos\theta\,\frac{d\theta}{dx} \\ dx = \cos\theta\, d\theta \\ \theta = \sin^{-1}x \end{array}\right.$

(iii) $\int \cos^{-1}x\,dx$

$= \int 1\,dx\,\cos^{-1}x - \int\left(\int 1\,dx\cdot\frac{d\cos^{-1}x}{dx}\right)dx$

$= x\cos^{-1}x - \int x\cdot\frac{-1}{\sqrt{1-x^2}}\,dx$

$= x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\cos^{-1}x + \int \frac{-dt}{2\sqrt{t}}$

$= x\cos^{-1}x - \frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$= x\cos^{-1}x - \frac{1}{2}\int dt \cdot t^{-\frac{1}{2}}$

$= x\cos^{-1}x - \frac{1}{2} \frac{t^{\frac{1}{2}}}{\frac{1}{2}}$

$= x\cos^{-1}x - \frac{1}{\cancel{2}} \cancel{2} \sqrt{t} + c = x\cos^{-1}x - \sqrt{1-x^2} + c$

Let $1-x^2 = t$

$-2x = \frac{dt}{dx}$

$x\,dx = \frac{dt}{-2}$

(iv) $\int x\tan^{-1}x\,dx$

$= \int x\,dx \tan^{-1}x - \int\left(\int x\,dt \frac{\tan^{-1}x}{dx}\right)dx$

$= \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx + \cancel{\frac{}{1+x^2}}$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{\tan^2\theta}{1+\tan^2\theta}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{\tan^2\theta \cancel{\sec^2\theta\,d\theta}\ \sec^2\theta\,d\theta}{\sec^2\theta}$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \tan^2\theta\,d\theta$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\sec\theta + c$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\sqrt{1+x^2} + c$$

$$\left\{ \begin{array}{l} \text{Let } x = \tan\theta \\ 1 = \sec^2\theta \frac{d\theta}{dx} \\ dx = d\theta\sec^2\theta\, d\theta \end{array} \right.$$

(v) $\int x^2 \tan^{-1}x\, dx$

$$= \int x^2 dx \cdot \tan^{-1}x - \int\left(\int x^2 dx \cdot \frac{d\tan^{-1}x}{dx}\right)dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^2 \cdot x}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{(x^2+1-1)}{1+x^2}x\,dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(\frac{x^2+1}{1+x^2}dx - \frac{1}{1+x^2}dx\right)x$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x\,dx - \int \frac{x}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3} \cdot \frac{x^2}{2} + \frac{1}{3}\int \frac{dt}{2t}$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\ln t + c$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + c$$

$$\left\{ \begin{array}{l} \text{Let } x^2 = t \\ 2x = \frac{dt}{dx} \Rightarrow x\,dx = \frac{dt}{2} \end{array} \right.$$

(vi) $\int \sec^{-1}x\,dx$

$= \int 1dx \cdot \sec^{-1}x - \int\left(\int 1dx \cdot \frac{d\sec^{-1}x}{dx}\right)\cdot dx$

$= x\sec^{-1}x - \int x \cdot \frac{1}{x\sqrt{x^2-1}}\cdot dx$

$= x\sec^{-1}x - \int \frac{dx}{\sqrt{x^2-1}}$

Let $x = \sec\theta$

$1 = \sec\theta \cdot \tan\theta \cdot \frac{d\theta}{dx}$

$dx = \sec\theta \cdot \tan\theta \cdot d\theta$

$= x\sec^{-1}x - \int \frac{\sec\theta \cdot \tan\theta \, d\theta}{\tan\theta}$

$= x\sec^{-1}x - \int \sec\theta\, d\theta$

$= x\sec^{-1}x - \ln|\sec\theta + \tan\theta| + c$

$= x\sec^{-1}x - \ln|x + \sqrt{x^2-1}| + c$

(vii) $\int x\cos^{-1}x\,dx$

$= \int x\,\mathrm{cosec}^{-1}x\,dx - \int\left(\int x \frac{d\,\mathrm{cosec}^{-1}x}{dx}\right)\cdot dx$

$= \frac{x^2}{2}\mathrm{cosec}^{-1}x - \int \frac{x^2}{2} \cdot \frac{-1}{x\sqrt{x^2-1}}dx$

$= \frac{x^2}{2}\mathrm{cosec}^{-1}x + \frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}dx$

$= \frac{x^2}{2}\mathrm{cosec}^{-1}x + \frac{1}{2}\int \frac{\mathrm{cosec}\theta \cdot \mathrm{cosec}\theta \cdot \cot\theta\, d\theta}{\cos\theta}$

$= \frac{x^2}{2}\operatorname{cosec}^{-1}x + \frac{1}{2}\int -\operatorname{cosec}^3\theta\, d\theta$

$= \frac{x^2}{2}\operatorname{cosec}^{-1}x - \frac{1}{2} - \cot\theta + c$

$= \frac{x^2}{2}\operatorname{cosec}^{-1}x + \frac{1}{2}\sqrt{x^2-1} + c$

let $x = \operatorname{cosec}\theta$
$\frac{dx}{d\theta} = -\operatorname{cosec}\theta\cot\theta$
$\therefore dx = -\operatorname{cosec}\theta\cot\theta\, d\theta$

5(i) $\int e^{3x}\cos 2x\, dx$

$= \int e^{3x}dx \cos 2x - \int\left(\int e^{3x}dx \cdot \frac{d\cos 2x}{dx}\right)dx$

$= \frac{1}{3}e^{3x}\cos 2x - \int \frac{1}{3}e^{3x}\cdot 2-\sin 2x \cdot dx$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x}\sin 2x\, dx$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\left[\int e^{3x}dx \cdot \frac{d\sin 2x}{dx}\, dx\right]$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\left[\frac{e^{3x}}{3}\sin 2x - \int \frac{e^{3x}}{3}\cdot 2\cos 2x\, dx\right]$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x - \frac{4}{9}\int e^{3x}\cos 2x\, dx$

$\therefore \int e^{3x}\cos 2x\, dx = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x - \frac{4}{9}\int e^{3x}\cos 2x\, dx$

$\Rightarrow \int e^{3x}\cos 2x\, dx\left(1+\frac{4}{9}\right) = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x$

$$\Rightarrow \int e^{3x}\cos 2x\,dx = \frac{1}{3}\frac{\left(e^{3x}\cos 2x + \frac{2}{3}e^{3x}\sin 2x\right)}{\frac{13}{9}}$$

$$= \frac{3e^{3x}\cos 2x + \frac{2}{3}e^{3x}\sin 2x}{13}$$

(ii) $\int e^{2x}\sin x\,dx$

$$= \int e^{2x}.\ \sin x\,dx - \int\left(\int e^{2x}.\ \frac{d\sin x}{dx}\right)dx$$

$$= \frac{1}{2}e^{2x}\sin x - \int \frac{e^{2x}}{2}\cos x\,dx$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left\{\int e^{2x}dx - \int\int e^{2x}\frac{d\cos x}{dx}dx\right\}$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x + \frac{1}{2}\int e^{2x}.\ -\sin x$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{2}\int e^{2x}\sin x\,dx$$

$$\therefore \int e^{2x}\sin x\,dx + \frac{1}{2}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x + C$$

$$\Rightarrow \frac{3}{2}\int e^{2x}\sin x\,dx = \frac{1}{2}\left(e^{2x}\sin x - \frac{1}{2}e^{2x}\cos x\right) + c$$

$$\Rightarrow \int e^{2x}\sin x\,dx = \frac{1}{3}\left(e^{2x}\sin x - \frac{1}{2}e^{2x}\cos x\right) + c$$

(iii) $\int e^x \cos^2 x \, dx$

$= \int e^x \frac{(1+\cos 2x)}{2} dx = \frac{1}{2}\int e^x (1+\cos 2x) dx$

$= \frac{1}{2}\left[\int e^x dx + \int e^x \cos 2x \, dx\right]$

$= \frac{1}{2} e^x + \frac{1}{2} I$ $\qquad \boxed{I = \int e^x \cos 2x \, dx}$

$I = \int e^x \cos 2x \, dx - \int\left(\int e^x \cdot \frac{d\cos 2x}{dx}\right) dx$

$= e^x \cos 2x + 2\int e^x \sin 2x \, dx$

$\frac{1}{2} e^x + \frac{1}{2}\left(e^x \cos 2x + 2\int e^x \sin 2x \, dx\right)$

$= \frac{1}{2} e^x + \frac{1}{2} e^x \cos 2x + \int e^x \sin 2x \, dx + c$

(iv) $\int x e^{x^2} \sin x^2 \, dx$

(v) $\int e^{ax} \sin(bx+c)\, dx$

$= \int e^{ax} \sin(bx+c)\, dx - \int\left(\int e^{ax} \frac{d \sin(bx+c)}{dx}\right) dx$

$= \frac{1}{a} e^{ax} \sin(bx+c) - \int \frac{1}{a} e^{ax} \cos(bx+c) \cdot b\, dx$

$= \frac{1}{a} e^{ax} \sin(bx+c) - \frac{b}{a} \int e^{ax} \cos(bx+c)\, dx$

$= \frac{1}{a} e^{ax} \sin(bx+c) - \frac{b}{a}\left[\int e^{ax} dx \cdot \cos(bx+c) - \int\left(\int e^{ax} dx \frac{d \cos(bx+c)}{dx}\right)\right]$

$= \frac{1}{a} e^{ax} \sin(bx+c) - \frac{b}{a^2} \int e^{ax} \cos(bx+c) - \frac{b^2}{a^2} \int e^{ax} \sin(bx+c) + c$

$= 2\int e^{ax} \sin(bx+c) = \frac{-b}{a^2}\left[\int e^{ax} \cos(bx+c) + b \int e^{ax} \sin(bx+c)\right] + c$

$= \int e^{ax}\sin(bx+c) = \dfrac{-\frac{b}{a^2}\left[\int e^{ax}\cos(bx+c) + b\int e^{ax}\sin(bx+c)\right]}{2}$

(vi) $\int (2x^2+1)\, e^{x^2} dx$

$= \int e^{x^2} dx \cdot (2x^2+1) - \int\left(\int e^{x^2} \frac{d(2x^2+1)}{dx} dx\right)$

$= e^{x^2}(2x^2+1) - \int e^{x^2}\cdot 4x\, dx$

$= e^{x^2}(2x^2+1) - \left[\int e^{x^2}\cdot 4x - \int\left(\int e^{x^2} \frac{d4x}{dx}\right) dx\right]$

$= e^{x^2}(2x^2+1) - e^{x^2}4x + 4\int e^{x^2} dx$

$= e^{x^2}(2x^2+1) - e^{x^2}4x + 4e^{x^2} + c$

$\therefore \int (2x^2+1)\, e^{x^2} dx = e^{x^2}\{2x^2+1-4x+4\}$

$= e^{x^2}\{2x^2-4x+5\} + c$

4/(i) $\int \sqrt{9-x^2}\, dx$

$= \int \sqrt{(3)^2 - x^2}\, dx$

$= \frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$

(ii) $\int \sqrt{5-9x^2}\, dx$

$$= \int \sqrt{(\sqrt{5})^2 - (2x)^2}\, dx$$

$$= \frac{2x}{2}\sqrt{5-4x^2} + \frac{5}{2}\sin^{-1}\frac{2x}{\sqrt{5}} + c$$

$$= x\sqrt{5-4x^2} + \frac{5}{2}\sin^{-1}\frac{2x}{\sqrt{5}} + c$$

(iii) $\int \sqrt{1-x^2-2x}\, dx$

Let $x+1 = z$

$1 = \frac{dz}{dx}$

$dx = dz$

$$= \int \sqrt{-(x^2+2x-1)}\, dx$$

$$= \int \sqrt{-(x^2+2\cdot x\cdot 1+(1)^2-(1)^2-1)}\, dx$$

$$= \int \sqrt{-\{(x+1)^2-2\}}\, dz$$

$$= \int \sqrt{-\{z^2-2\}}\, dz$$

$$= \int \sqrt{2-z^2}\, dz = \int \sqrt{(\sqrt{2})^2 - z^2}\, dx$$

$$= \frac{z}{2}\sqrt{2-z^2} + \frac{2}{2}\sin^{-1}\frac{z}{\sqrt{2}} + c$$

$$= \frac{x+1}{2}\sqrt{2-(x+1)^2} + \sin^{-1}\frac{x+1}{\sqrt{2}} + c$$

(iv) $\int e^x\sqrt{4-e^{2x}}\, dx$

Let $e^x = t$

$e^x = \frac{dt}{dx}$

$e^x\, dx = dt$

$$= \int e^x\sqrt{(2)^2-(e^x)^2}\, dx$$

$$= \int \sqrt{(2)^2 - t^2}\, dt$$

$= \frac{t}{2}\sqrt{4-t^2} + \frac{4}{2}\sin^{-1}\frac{t}{2} + C$

$= \frac{e^x}{2}\sqrt{4-e^{2x}} + 2\sin^{-1}\frac{e^x}{2} + C$

(3) $\int \cos\theta \sqrt{5-\sin^2\theta}\, d\theta$ $\left(\begin{array}{l} \text{let } \sin\theta = t \\ \cos\theta = \frac{dt}{d\theta} \\ \cos\theta\, d\theta = dt \end{array} \right.$

$= \int \sqrt{5-t^2}\, dt$

$= \frac{t}{2}\sqrt{5-t^2} + \frac{5}{2}\sin^{-1}\frac{t}{\sqrt{5}} + C$

$= \frac{\sin\theta}{2}\sqrt{5-\sin^2\theta} + \frac{5}{2}\sin^{-1}\left(\frac{\sin\theta}{\sqrt{5}}\right) + C$

7/(i) $\int \sqrt{x^2+4}\, dx$

$= \int \sqrt{x^2+(2)^2}\, dx$

$= \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\ln\left|x+\sqrt{x^2+4}\right| + C$

$= \frac{x}{2}\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| + C$

(ii) $\int \sqrt{7x^2+2}\, dx$ $\left(\begin{array}{l} \text{let } \sqrt{7}x = t \\ \sqrt{7} = \frac{dt}{dx} \\ dx = \frac{dt}{\sqrt{7}} \end{array} \right.$

$= \int \sqrt{(\sqrt{7}x)^2+(\sqrt{2})^2}\, dx$

$= \int \sqrt{t^2+(\sqrt{2})^2}\cdot\frac{dt}{\sqrt{7}}$

$= \frac{1}{\sqrt{7}}\left[\frac{x}{2}\sqrt{7x^2+2} + \frac{2}{2}\ln\left|\sqrt{7}x+\sqrt{7x^2+2}\right|\right] + C$

(iii) $\int \sqrt{4x^2+12x+13}\, dx \quad (2x+3=z)$

$= \int \sqrt{(2x)^2 + 2.(2x).3 + (3)^2 - (3)^2 + 13}\, dx$

$= \int \sqrt{(2x+3)^2 - 9 + 13}\, dx$

$= \int \sqrt{(2x+3)^2 + 4}\, dx$

$= \frac{2x+3}{2}\sqrt{(2x+3)^2+4} + \frac{4}{2} \ln \left| 2x+3 + \sqrt{(2x+3)^2+4} \right| + c$

$= \frac{2x+3}{2}\sqrt{(2x+3)^2+4} + \frac{\cancel{4}}{2} \ln \left| 2x+3 + \sqrt{(2x+3)^2+4} \right| + c$

(iv) $\int e^{2z} \sqrt{e^{4z}+6}\, dz$

$= \int e^{2z} \sqrt{(e^{2z})^2+6}\, dz$

Let $e^{2z} = t$

$2e^{2z} = \frac{dt}{dz}$

$e^{2z} dz = \frac{dt}{2}$

$= \int \sqrt{t^2+6}\, \frac{dt}{2}$

$= \frac{1}{2}\int \sqrt{t^2+6}\, dt$

$= \frac{1}{2} \cdot \frac{t}{2}\sqrt{t^2+6} + \frac{\sqrt{6}}{2} \ln \left| t + \sqrt{t^2+6} \right| + c$

$= \frac{e^{2z}}{4}\sqrt{e^{4z}+6} + \sqrt{\frac{6}{\cancel{2}}} \ln \left| e^{2z} + \sqrt{e^{4z}+6} \right| + c$

$= \frac{1}{4}\left[e^{2z}\sqrt{e^{4z}+6} + \frac{\sqrt{6}}{2} \ln \left| e^{2z} + \sqrt{e^{4z}+6} \right| \right] + c$

(v) $\int \sec^2\theta \sqrt{\sec^2\theta+3}\, d\theta$

$= \int \sec^2\theta \sqrt{(1+\tan^2\theta)+3}\, d\theta$

$= \int \sec^2\theta\, d\theta \sqrt{1+z^2+3}$ $\left[\begin{array}{l} \text{let } \tan\theta = z \\ \sec^2\theta = \frac{dz}{d\theta} \\ \sec^2\theta\, d\theta = dz \end{array}\right.$

$= \int \sqrt{z^2+4}\, dz$

$= \frac{z}{2}\sqrt{z^2+4} + \frac{4}{2}\ln\left|z+\sqrt{z^2+4}\right| + C$

$= \frac{\tan\theta}{2}\sqrt{\tan^2\theta+4} + 2\ln\left|\tan\theta+\sqrt{\tan^2\theta+4}\right| + C$

$= \frac{1}{2}\left[\tan\theta\sqrt{\tan^2\theta+4} + 4\ln\left|\tan\theta+\sqrt{\tan^2\theta+4}\right|\right] + C$

$= \frac{1}{2}\left[\tan\theta\sqrt{\sec^2\theta+3} + 4\ln\left|\tan\theta+\sqrt{\sec^2\theta+3}\right|\right] + C$

(vi) $\int (2x^2+1)e^{x^2}\, dx$

8(i) $\int \sqrt{x^2-8}\,dx$

$= \int \sqrt{x^2-(\sqrt{8})^2}\,dx$

$= \frac{x}{2}\sqrt{x^2-8} - \frac{8}{2}\ln\left|x+\sqrt{x^2-8}\right| + c$

$= \frac{x}{2}\sqrt{x^2-8} - 4\ln\left|x+\sqrt{x^2-8}\right| + c$

(ii) $\int \sqrt{3x^2-2}\,dx$

$= \int \sqrt{(\sqrt{3}x)^2-(\sqrt{2})^2}\,dx$

$= \frac{x}{2}\sqrt{3x^2-2} - \frac{2}{2}\ln\left|\sqrt{3}x+\sqrt{3x^2-2}\right| + c$

$= \frac{x}{2}\sqrt{3x^2-2} - \ln\left|\sqrt{3}x+\sqrt{3x^2}\right| + c$

(iii) $\int \sqrt{x^2-4x+2}\,dx \quad (x-2=z)$

$= \int \sqrt{x^2 - 2\cdot x\cdot 2 + (2)^2 - (2)^2 + 2}\, dx$

$= \int \sqrt{(x-2)^2 - 2}\, dx$ $\left[\begin{array}{l} \text{let } x - 2 = t \\ 1 = \frac{dt}{dx} \\ dx = dt \end{array}\right.$

$= \int \sqrt{t^2 - 2}\, dt$

$= \frac{t}{2}\sqrt{t^2-2} - \frac{2}{2}\ln\left|t + \sqrt{t^2 - 2}\right| + c$

$= \frac{x-2}{2}\sqrt{(x-2)^2 - 2} - \ln\left|x - 1\sqrt{(x-1)^2 - 2}\right| + c$

$= \frac{x-2}{2}\sqrt{x^2 - 4x + 2} - \ln\left|x - 1\sqrt{x^2 - 4x + 2}\right| + c$

(iv) $\int a^2 \sqrt{a^4 - 4}\, da$ $\left[\begin{array}{l} \text{let } a = t \\ a^2 = \frac{dt}{da} \\ a^2 da = dt \end{array}\right.$

$= \int \sqrt{(a^2)^2 - (2)^2}\, a^2 da$

$= \int \sqrt{t^2 - (2)^2}\, dt$

$= \frac{t}{2}\sqrt{4 - a^2 2} + \frac{4}{2}\ln\left|t + \sqrt{t^2 + 4}\right| + c$

$= \frac{a^2}{2}\sqrt{4 - a^2 2} + 2\ln\left|a^2 + \sqrt{a^2 2 + 4}\right| + c$

(v) $\int \sec\theta \tan\theta \sqrt{\tan^2\theta - 3}\, d\theta$

$\left[\begin{array}{l} \text{let } \sec\theta = t \\ \sec\theta \tan\theta = \frac{dt}{d\theta} \\ \sec\theta \tan\theta\, d\theta = dt \end{array}\right.$

$= \int \sec\theta \cdot \tan\theta \sqrt{\sec^2\theta - 1 - 3}\, d\theta$

$= \int \sec\theta \tan\theta \sqrt{\sec^2\theta - 4}\, d\theta$

$= \int \sqrt{t^2 - 4}\, dt$

$= \frac{t}{2}\sqrt{t^2 - 4} - \frac{4}{2}\ln\left|t + \sqrt{t^2 - 4}\right| + c$

$= \frac{1}{2}\sec\theta \sqrt{\tan^2\theta - 3} - 2\ln\left|\sec\theta + \tan^2\theta - 3\right| + c$

9 (i) $\int e^x(\tan x + \ln \sec x)\,dx$

$= \int f'(x)\,dx$ Let $f(x) = e^x \ln \sec x$

$= f(x) + C$ $f'(x) = e^x \ln \sec x + \tan x \cdot e^x$

$= e^x \ln \sec x + c$ $= e^x(\ln \sec x + \tan x)$

(ii) $\int e^x(\cot x + \ln \sin x)\,dx$ Let

$= \int f'(x)\,dx$ $f(x) = e^x \ln \sin x$

$= f(x) + c$ $f'(x) = e^x \ln \sin x + \frac{1}{\sin x}\cdot \cos x \cdot e^x$

$= e^x \ln \sin x + c$ $= e^x(\ln \sin x + \cot x)$

(iii) $\int \frac{e^x}{x}(1 + x \ln x)\,dx$

$= \int e^x\left(\frac{1}{x} + \ln x\right)dx$

$= \int f'(x)\,dx$

$= f(x) + c$

$= e^x \ln x + c$

(iv) $\int \frac{x e^x}{(1+x)^2}\,dx$ Let $f(x) = e^x \frac{1}{(1+x)}$

$= \int e^x \frac{x+1-1}{(1+x)^2}\,dx$ $f'(x) = \frac{e^x(1+x) - 1\cdot e^x}{(1+x)^2}$

$= \int e^x\left(\frac{(x+1)}{(1+x)^2} - \frac{1}{(1+x)^2}\right)dx$ $= e^x\left(\frac{1}{1+x} - \frac{1}{(1+x)^2}\right)$

$= \int e^x \left(\frac{1}{x+1} - \frac{1}{(1+x)^2}\right) dx$

$= \int f'(x)\, dx$

$= f(x) + c = \frac{e^x}{1+x} + c$

$~~19~~$ (i) $\int \left[\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right] dx$

$= \int (\ln x)^{-1} dx - \int (\ln x)^{-2} dx$

$= \left(\int 1\, dx\right)(\ln x)^{-1} - \int \left(\int 1\, dx\right) \frac{d(\ln x)^{-1}}{dx} dx - \int (\ln x)^2 dx$

$= x(\ln x)^{-1} - \int x(-1)(\ln x)^{-2} \frac{1}{x} dx - \int (\ln x)^{-2} dx$

$= x(\ln x)^{-1} + \int (\ln x)^{-2} dx - \int (\ln x)^{-2} dx$

$= x(\ln x)^{-1} + c$

(ii) $\int \sin(\ln x)\, dx$

$= \int 1\, dx \sin(\ln x) - \int \left(\int 1\, dx\right) \frac{d \sin(\ln x)}{dx} dx$

$= x \sin(\ln x) - \int x \cos \ln x \cdot \frac{1}{x} dx$

$= x \sin(\ln x) - \left[\int (1\, dx) \cos \ln x - \int \left(\int 1\, dx\right) \frac{d \cos(\ln x)}{dx} dx\right]$

$= x\sin(\ln x) - \left[x\cos\ln x - \int x \cdot (-\sin(\ln x))\frac{1}{x}dx\right]$

$= x\sin(\ln x) - x\cos\ln x - \int \sin \ln x \, dx$

$\therefore \int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos\ln x - \int \sin\ln x\,dx$

$\Rightarrow 2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos\ln x$

$\Rightarrow \int \sin(\ln x)\,dx = \frac{x}{2}(\sin\ln x - \cos\ln x) + c$

(iii) $\int \sin x \ln(\operatorname{cosec} x - \cot x)\,dx$

$= \int (\sin x)(\ln(\operatorname{cosec} x - \cot x))\,dx$

$\int \left(\int \sin x \, dx \frac{d \ln(\operatorname{cosec} x - \cot x)}{dx}\right) dx$

$= -\cos x \ln(\operatorname{cosec} x - \cot x) - \int -\cos x \cdot \frac{1}{\operatorname{cosec} x - \cot x} \cdot (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)$

$= -\cos x \ln(\operatorname{cosec} x - \cot x) + \int \frac{\cos x \cdot \operatorname{cosec} x}{\operatorname{cosec} x - \cot x} \frac{(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}\,dx$

$= -\cos x \ln(\operatorname{cosec} x - \cot x) + \int \cot x$

$= -\cos x \ln(\operatorname{cosec} x - \cot x) + \ln|\sin x| + c$

$$= \int e^x \left(\frac{1}{x+1} - \frac{1}{(1+x)^2}\right) dx$$

$$= \int f'(x)\, dx$$

$$= f(x) + c = \frac{e^x}{1+x} + c$$

~~10~~ (i) $\int \left[\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right] dx$

$$= \int (\ln x)^{-1} dx - \int (\ln x)^{-2} dx$$

$$= \left(\int 1\, dx\right)(\ln x)^{-1} - \int \left(\int 1\, dx\right) \frac{d(\ln x)^{-1}}{dx}\, dx - \int (\ln x)^{-2} dx$$

$$= x(\ln x)^{-1} - \int x(-1)(\ln x)^{-2} \frac{1}{x} dx - \int (\ln x)^{-2} dx$$

$$= x(\ln x)^{-1} + \int (\ln x)^{-2} dx - \int (\ln x)^{-2} dx$$

$$= x(\ln x)^{-1} + c$$

(ii) $\int \sin(\ln x)\, dx$

$$= \int 1\, dx \sin(\ln x) - \int \left[\int 1\, dx\right] \cdot \frac{d \sin(\ln x)}{dx} dx$$

$$= x \sin(\ln x) - \int x \cos \ln x \cdot \frac{1}{x} dx$$

$$= x \sin(\ln x) - \left[\int (1\, dx) \cos \ln x - \int \left(\int 1\, dx\right) \frac{d \cos(\ln x)}{dx} dx\right]$$

$= x\sin(\ln x) - \left[x\cos\ln x - \int x\cdot(-\sin(\ln x))\frac{1}{x}\right]$

$= x\sin(\ln x) - x\cos\ln x - \int \sin\ln x\,dx$

$\therefore \int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos\ln x - \int \sin\ln x\,dx$

$\Rightarrow 2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos\ln x$

$\Rightarrow \int \sin(\ln x)\,dx = \frac{x}{2}(\sin\ln x - \cos\ln x) + c$

(iii) $\int \sin x \ln(\operatorname{cosec} x - \cot x)\,dx$

$= \int (\sin x).(\ln(\operatorname{cosec} x - \cot x))\,dx$

$\int \left(\int \sin x \cdot \frac{d\ln(\operatorname{cosec} x - \cot x)}{dx}\right)dx$

$= -\cos x\ln(\operatorname{cosec} x - \cot x) - \int -\cos x \cdot \frac{1}{\operatorname{cosec} x - \cot x}\cdot(-\operatorname{cosec} x\cot x + \operatorname{cosec}^2 x)$

$= -\cos x\ln(\operatorname{cosec} x - \cot x) + \int \frac{\cos x\cdot\operatorname{cosec} x\,\frac{(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}}{\operatorname{cosec} x - \cot x}\,dx$

$= -\cos x\ln(\operatorname{cosec} x - \cot x) + \int \cot x$

$= -\cos x\ln(\operatorname{cosec} x - \cot x) + \ln|\sin x| + c$

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3(v) $\int x \sin^3 x \, dx$

$= \int x \cdot \left[\frac{3\sin x - \sin 3x}{4}\right] dx$

$= \int x \left[\frac{3}{4}\sin x - \frac{\sin 3x}{4}\right] dx$

$= \int \left[\frac{3x}{4}\sin x - \frac{x}{4}\sin 3x\right] dx$

$= \int \frac{3x}{4}\sin x \, dx - \int \frac{x}{4}\sin 3x \, dx$

$= \frac{3}{4}\int x \sin x \, dx - \frac{1}{4}\int x \sin 3x \, dx$

$= \frac{3}{4}\left[\int \sin x \, dx \cdot x - \int\left(\int \sin x \, dx\right)\cdot \frac{dx}{dx} \cdot dx\right]$

$- \frac{1}{4}\left[\int \sin 3x \, dx \cdot x - \int\left(\int \sin 3x \, dx \cdot \frac{dx}{dx} \cdot dx\right)\right]$

$= \frac{3}{4}\left[-\cos x \cdot x \int -\cos x \, dx\right]$

$- \frac{1}{4}\left[-\frac{\cos 3x}{3} \cdot x - \int -\frac{\cos 3x}{3} \cdot dx\right]$

$= -\frac{3x}{4}\cos x + \frac{3}{4}\int \cos x \, dx$

$+ \frac{1}{12}\cos 3x \cdot x - \frac{1}{4}\int \frac{\cos 3x}{3} dx$

$= -\frac{3x}{4}\cos x + \frac{3}{4}\sin x + \frac{x}{12}\cos 3x$

$- \frac{1}{12} \times \frac{1}{3}\sin 3x + c$

$= \frac{1}{4}\left[-3x\cos x + 3\sin x + \frac{1}{3}\cos 3x - \frac{1}{9}\sin 3x\right] + c$

3(vi) $\int \ln(x^2+x+2) \, dx$

$= \int 1 \cdot \ln(x^2+x+2) \, dx$

$= \int 1\,dx \cdot \ln(x^2+x+2) - \int \left(\int 1\,dx\right) \frac{d}{dx} \ln(x^2+x+2)\,dx$

$= x\ln(x^2+x+2) - \int x \cdot \frac{d\ln(x^2+x+2)}{d(x^2+x+2)} \cdot \frac{d(x^2+x+2)}{dx}$

$= x\ln(x^2+x+2) - \int x \cdot \frac{1}{x^2+x+2}(2x+1) \cdot dx$

$= x\ln(x^2+x+2) - \int \frac{2x^2+x}{x^2+x+2}\,dx$

$= x\ln(x^2+x+2) - 2\int \frac{x^2+\frac{x}{2}}{x^2+x+2}\,dx$ --- (1)

Now, $\int \frac{x^2+\frac{x}{2}}{x^2+x+2}\,dx$

$= \int \frac{x^2+x+2-x-2+\frac{x}{2}}{x^2+x+2}\,dx$

$= \int \frac{x^2+x+2}{x^2+x+2}\,dx + \int \frac{\left(-x-2+\frac{x}{2}\right)}{x^2+x+2}\,dx$

$= \int 1\,dx + \int \left(\frac{\frac{-2x-4+x}{2}}{x^2+x+2}\right)dx$

$= x + \frac{1}{2}\int \frac{-x-4}{x^2+x+2}\,dx$

$= x + \frac{1}{2} - \left(\int \frac{x+4}{x^2+x+2}\,dx\right)$

$= x - \frac{1}{2}\int \frac{x+4}{x^2+x+2}\,dx$

$= x - \frac{1}{2} \cdot \frac{1}{2} \cdot 2 \int \frac{x+4}{x^2+x+2} dx$

$= x - \frac{1}{4} \int \frac{2x+8}{x^2+x+2} dx$

$= x - \frac{1}{4} \int \frac{2x+1+7}{x^2+x+2} dx$

$= x - \frac{1}{4} \left[\int \frac{2x-1}{x^2+x+2} dx + 7 \int \frac{dx}{x^2+x+2} \right]$

$= x - \frac{1}{4} \ln |x^2+x+2| - \frac{7}{4} \int \frac{dx}{x^2 + 2 \cdot \frac{1}{2} \cdot x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 2}$

$= x - \frac{1}{4} \ln |x^2+x+2| - \frac{7}{4} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} dx$

$= x - \frac{1}{4} \ln |x^2+x+2| - \frac{7}{4} \cdot \frac{1}{\frac{\sqrt{7}}{2}} \cdot \tan^{-1} \left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) + C$

$= x - \frac{1}{4} \ln |x^2+x+2| - \frac{\cancel{7}^{\sqrt{7}}}{\cancel{4}_{2}} \cdot \frac{\cancel{2}}{\sqrt{7}} \cdot \tan^{-1} \left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) + C$

$\therefore$ eqn (i) becomes,

$= x \ln |x^2+x+2| - 2x + \frac{1}{2} \ln |x^2+x+2| + \sqrt{7} \tan^{-1} \frac{2x+1}{\sqrt{7}} + C$

4(ii) $\int x \sin^{-1} x \, dx$

$= \int x \, dx \sin^{-1} x - \int \left(\int x \, dx \cdot \frac{d \sin^{-1} x}{dx} \right) dx$

$= \frac{x^2}{2} \sin^{-1} x - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx$

$$= \frac{x^2 \sin^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2 \sin^{-1}x}{2} - \frac{1}{2}\int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2 \sin^{-1}x}{2} - \frac{1}{2}\left[\int \frac{-(1-x^2)}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$= \frac{x^2 \sin^{-1}x}{2} + \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\sin^{-1}x + c$$

Let $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$\frac{dx}{dx} = \frac{d\sin\theta}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\Rightarrow 1 = \cos\theta \cdot \frac{d\theta}{dx}$$

$$\Rightarrow dx = \cos\theta\, d\theta$$

$$= \frac{x^2 \sin^{-1}x}{2} - \frac{1}{2}\sin^{-1}x + \frac{1}{2}\int \sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{2}\int \cos\theta\cdot\cos\theta\, d\theta$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{2}\int \cos^2\theta\, d\theta$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{2}\cdot\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{4}\int (1+\cos 2\theta)\, d\theta$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{4}\left[\int 1\, d\theta + \int \cos 2\theta\, d\theta\right]$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{4}\cdot\left[\theta + \frac{1}{2}\sin\theta\right]$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + c$$

$$= (x^2-1)\frac{\sin^{-1}x}{2} + \frac{1}{4}\sin^{-1}x + \frac{1}{8}\sin 2\sin^{-1}x + c$$